

Symmetries of the dual metrics

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Abstract

In this paper the symmetries of the dual manifold were investigated. We found the conditions when the manifold and its dual admit the same Killing vectors and Killing-Yano tensors. In the case of an Einstein's metric $g_{\mu\nu}$ the corresponding equations for its dual were found. The examples of Kerr-Newman geometry and the separable coordinates in $1+1$ dimensions were analyzed in details.

1 Introduction

In a geometrical setting, symmetries are connected with isometries associated with Killing vectors, and more generally, with Killing tensors on the configuration space of the system. An example is the motion of a point particle in a space with isometries which is a physicist's way of studying the geodesic structure of a manifold [1]. Such studies were extended to spinning space-times described by supersymmetric extensions of the geodesic motion [2] and it was shown that this can give rise to interesting new types of supersymmetry as well. The "non-generic" symmetries were investigated in the case of Taub-NUT metric [3] and extended Taub-NUT metric [4]. It was a big success of Gibbons et al. [2] to have been able to show that the Killing-Yano tensor [5], which had long been known to relativists as a rather mysterious structure, can be understood as an object generating a "non-generic" supersymmetry, i.e a supersymmetry appearing only in specific space-times. Killing tensors are important for solving the equations of motion in particular space-times. The notable example here is the Kerr metric which admits a second rank Killing tensor [2].

In [6] a new geometric duality was introduced but the physical interpretation of the dual metrics was not yet clarified. Let us consider a space with metric $g_{\mu\nu}$ admitting a Killing tensor field $K_{\mu\nu}$ [6]. The equation of motion of a particle on a geodesic is derived from the action $S = \int d\tau (\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)$. The Hamiltonian is constructed in the form $H = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu$ and the Poisson brackets are $\{x_\mu, p^\nu\} = \delta_\mu^\nu$. The equation of motion for a phase space function $F(x, p)$ can be computed from the Poisson brackets with the Hamiltonian $\dot{F} = \{F, H\}$, where $\dot{F} = \frac{dF}{d\tau}$. From the covariant component $K_{\mu\nu}$ of the Killing tensor we

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can construct a constant of motion K , with $K = \frac{1}{2}K_{\mu\nu}p^\mu p^\nu$. It can be easily verified that $\{H, K\} = 0$. The formal similarity between the constants of motion H and K , and the symmetrical nature of the condition implying the existence of the Killing tensor amount to a reciprocal relation between two different models: the model with Hamiltonian H and constant of motion K , and a model with constant of motion H and Hamiltonian K . The relation between the two models has a geometrical interpretation: it implies that if $K_{\mu\nu}$ are the contravariant components of a Killing tensor with respect to the metric $g_{\mu\nu}$, then $g_{\mu\nu}$ must represent a Killing tensor with respect to the metric defined by $K_{\mu\nu}$. When $K_{\mu\nu}$ has an inverse we interpret it as the metric of another space and we can define the associated Riemann-Christoffel connection $\hat{\Gamma}_{\mu\nu}^\lambda$ as usual through the metric postulate $\hat{D}_\lambda K_{\mu\nu} = 0$, where \hat{D} represents the covariant derivative with respect to $K_{\mu\nu}$.

Recently, Killing tensors of third rank in $1 + 1$ dimensional geometry were classified [7] and the Lax tensors on the dual metric were investigated [8]. The non-standard Dirac operators which differ from, but commute with the standard Dirac operator, were analyzed on a manifold with non-trivial Killing tensor admitting a square root of Killing-Yano type [9].

For these reasons the symmetries of the dual manifolds are interesting to investigate.

The plan of this paper is as follows:

In Section 2 the symmetries of a dual manifold were investigated. In Section 3 two examples were analyzed. Our conclusions were presented in Section 4.

2 Dual metrics symmetries

A Killing tensor is a symmetric tensor which satisfies the following relation:

$$D_\lambda K_{\mu\nu} + D_\mu K_{\nu\lambda} + D_\nu K_{\lambda\mu} = 0, \quad (1)$$

where D represent the covariant derivative with respect $g_{\mu\nu}$.

If the Killing tensor is non-degenerate one can find the associate Riemann-Christoffel of the dual manifold

$$\hat{\Gamma}_{\mu\nu}^\lambda = \frac{1}{2}K^{\lambda\tau}\left(\frac{\partial K_{\mu\tau}}{\partial x^\mu} + \frac{\partial K_{\nu\tau}}{\partial x^\mu} - \frac{\partial K_{\mu\nu}}{\partial x^\tau}\right). \quad (2)$$

If the manifold is torsion free, taking into account (1) and (2), we found

$$\hat{\Gamma}_{\nu\lambda}^\mu = \Gamma_{\nu\lambda}^\mu - K^{\mu\delta}D_\delta K_{\nu\lambda}, \quad (3)$$

where $\Gamma_{\mu\nu}^\lambda$ is the associate Riemann-Christoffel connection with respect to $g_{\mu\nu}$.

We know that the conformal transformation of $g_{\mu\nu}$ is defined as

$$\hat{g}_{\mu\nu} = e^{2U} g_{\mu\nu}$$

$$\hat{g}^{\mu\nu} = e^{-2U} g^{\mu\nu}, U = U(x). \quad (4)$$

Using (4) the relation between the corresponding connections becomes

$$\hat{\Gamma}_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} + 2\delta_{(\mu}^{\lambda} U_{\nu)'} - g_{\mu\nu} U'^{\lambda}, \quad (5)$$

where $U'^{\lambda} = \frac{dU^{\lambda}}{dx}$. From (3) and (5) we conclude that the dual transformation (3) is not of a conformal transformation type.

For this reason it would be interesting to investigate which are the conditions when the manifold admits the same Killing vectors and Killing-Yano tensors as its dual manifold.

Let us denote by χ_{μ} a Killing vector corresponding to $g_{\mu\nu}$ and let $\hat{\chi}_{\mu}$ be a Killing vector corresponding to $K_{\mu\nu}$.

Proposition 1

The manifold $g_{\mu\nu}$ and its dual have the same Killing vector iff

$$(D_{\delta} K_{\mu\nu}) \hat{\chi}^{\delta} = 0. \quad (6)$$

Proof.

Since χ_{σ} is a Killing vector we have

$$D_{\mu} \chi_{\nu} + D_{\nu} \chi_{\mu} = 0. \quad (7)$$

In the dual space using (3) we found the following corresponding equations

$$D_{\mu} \hat{\chi}_{\nu} + D_{\nu} \hat{\chi}_{\mu} + 2K^{\delta\sigma} (D_{\delta} K_{\mu\nu}) \hat{\chi}_{\sigma} = 0, \quad (8)$$

where D is the covariant derivative on manifold $g_{\mu\nu}$.

If $\hat{\chi}_{\mu} = \chi_{\mu}$, then from (7) and (8) we get

$$(D_{\delta} K_{\mu\nu}) \hat{\chi}^{\delta} = 0. \quad (9)$$

Conversely, if (6) is satisfied, then from (8) we deduce immediately $\chi_{\mu} = \hat{\chi}_{\mu}$.
q.e.d.

A Killing-Yano $f_{\mu\nu}$ is an antisymmetric tensor which satisfies the following equations [5]

$$D_{\lambda} f_{\mu\nu} + D_{\mu} f_{\lambda\nu} = 0. \quad (10)$$

Let us suppose that a manifold $g_{\mu\nu}$ has the Killing-Yano tensor $f_{\mu\nu}$ and its dual admits the Killing-Yano tensor $\hat{f}_{\mu\nu}$.

Proposition 2

The manifold $g_{\mu\nu}$ and its dual have the same Killing-Yano tensor iff

$$\hat{f}_{\nu\delta} K^{\delta\sigma} D_{\sigma} K_{\mu\lambda} + 2\hat{f}_{\sigma\lambda} K^{\sigma\delta} D_{\delta} K_{\mu\nu} + \hat{f}_{\mu\sigma} K^{\sigma\delta} D_{\delta} K_{\nu\lambda} = 0. \quad (11)$$

Proof.

Taking into account (3) Killing-Yano equations in the dual space become

$$\begin{aligned}\hat{D}_\mu \hat{f}_{\nu\lambda} + \hat{D}_\nu \hat{f}_{\mu\lambda} &= D_\mu \hat{f}_{\nu\lambda} + D_\nu \hat{f}_{\mu\lambda} + \hat{f}_{\nu\delta} K^{\delta\sigma} D_\sigma K_{\mu\lambda} + 2\hat{f}_{\sigma\lambda} K^{\sigma\delta} D_\delta K_{\mu\nu} \\ &+ \hat{f}_{\mu\sigma} K^{\sigma\delta} D_\delta K_{\nu\lambda} = 0.\end{aligned}\quad (12)$$

If $\hat{f}_{\mu\nu} = f_{\mu\nu}$, using (12) and because $f_{\mu\nu}$ is a Killing-Yano tensor on $g_{\mu\nu}$ we conclude that (11) is identically zero.

Conversely, if (11) is satisfied then from (12) and (10) we have $\hat{f}_{\mu\nu} = f_{\mu\nu}$ q.e.d.

Remarks

i) The associated metric

$$K_{\mu\nu} = f_{\mu\lambda} f_\nu^\lambda \quad (13)$$

has the inverse if $f_{\mu\lambda}$ is non-degenerate.

ii) If the manifold admits two non-degenerate Killing-Yano tensors $f_{\mu\nu}$ and $F_{\mu\nu}$ then the dual metric has the form $K_{\mu\nu} = f_{\mu\lambda} F_\nu^\lambda + f_{\nu\lambda} F_\mu^\lambda$

An important question is if the dual metric satisfies the Einstein's equations. For this reason is interesting to calculate the connection between Riemann curvature tensor, Ricci tensor, Ricci scalar of the manifold $g_{\mu\nu}$ and the corresponding expressions of the dual manifold.

We know that

$$R_{\nu\rho\sigma}^\beta = \Gamma_{\nu\sigma,\rho}^\beta - \Gamma_{\nu\rho,\sigma}^\beta + \Gamma_{\nu\sigma}^\alpha \Gamma_{\alpha\rho}^\beta - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\sigma}^\beta. \quad (14)$$

Using (3) the corresponding dual Riemann curvature tensor $\hat{R}_{\nu\rho\sigma}^\beta$ has the following expression

$$\hat{R}_{\nu\rho\sigma}^\beta = R_{\nu\rho\sigma}^\beta + R_{\nu\rho\sigma}'^\beta, \quad (15)$$

where $R_{\nu\rho\sigma}'^\beta$ is

$$\begin{aligned}R_{\nu\rho\sigma}'^\beta &= -(K^{\beta\delta} D_\delta K_{\nu\sigma})_{,\rho} + (K^{\beta\chi} D_\chi K_{\nu\rho})_{,\sigma} - \Gamma_{\nu\sigma}^\alpha K^{\beta\chi} D_\chi K_{\alpha\rho} \\ &- \Gamma_{\alpha\rho}^\beta K^{\alpha\delta} D_\delta K_{\nu\sigma} + \Gamma_{\nu\rho}^\alpha K^{\beta\chi} D_\chi K_{\alpha\sigma} + \Gamma_{\alpha\sigma}^\beta K^{\alpha\delta} D_\delta K_{\nu\rho} \\ &- K^{\alpha\delta} D_\delta K_{\nu\sigma} K^{\beta\chi} D_\chi K_{\alpha\rho} + K^{\alpha\delta} D_\delta K_{\nu\rho} K^{\beta\chi} D_\chi K_{\alpha\sigma}.\end{aligned}\quad (16)$$

The explicit expression of the Ricci tensor becomes

$$R_{\mu\nu} = \Gamma_{\mu\alpha,\nu}^\alpha - \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta + \Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta, \quad (17)$$

then the dual Ricci tensor becomes

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + R_{\mu\nu}'. \quad (18)$$

Here

$$R_{\mu\nu}' = -(K^{\alpha\chi} D_\chi K_{\mu\alpha})_{,\nu} + (K^{\alpha\chi} D_\chi K_{\mu\nu})_{,\alpha} - \Gamma_{\mu\nu}^\alpha K^{\beta\delta} D_\delta K_{\alpha\beta}$$

$$\begin{aligned}
& - \Gamma_{\mu\beta}^{\alpha} K^{\beta\delta} D_{\delta} K_{\nu\alpha} - \Gamma_{\alpha\beta}^{\beta} K^{\alpha\chi} D_{\chi} K_{\mu\nu} - \Gamma_{\nu\alpha}^{\beta} K^{\alpha\chi} D_{\chi} K_{\mu\beta} \\
& - K^{\beta\delta} K^{\alpha\chi} D_{\chi} K_{\mu\nu} D_{\delta} K_{\alpha\beta} + K^{\alpha\chi} D_{\chi} K_{\mu\beta} K^{\beta\delta} D_{\delta} K_{\nu\alpha}.
\end{aligned} \tag{19}$$

If the metric $g_{\mu\nu}$ satisfies Einstein's equations in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{20}$$

then the dual metric $K_{\mu\nu}$ satisfies the following equation

$$(\hat{R}_{\lambda\sigma} - R'_{\lambda\sigma})(\delta_{\lambda\mu}\delta_{\sigma\nu} - \frac{1}{2} g_{\mu\nu} g^{\lambda\sigma}) = 0, \tag{21}$$

where $\hat{R}_{\mu\nu}$ and $R'_{\mu\nu}$ are given by (18) and (19). Using Eq.(21) we found that $K_{\mu\nu}$ is not an Einstein's metric. The similar results can be obtained also in the presence of matter and cosmological constant. On the other hand if $R_{\mu\nu} = 0$, taking into account (18) we can deduce that $R'_{\mu\nu}$ is non-zero. If the manifold $g_{\mu\nu}$ has constant scalar curvature then the scalar curvature of dual manifold is not constant.

In the case of the Euclidean flat space, for example, we have a Killing-Yano tensor of order two $f_{\mu\nu} = a_{\mu\nu\lambda} x^{\lambda}$ which generate a Killing tensor $K_{\mu\nu}$ as $K_{\mu\nu} = a_{\mu\gamma\lambda} a_{\nu\sigma}^{\gamma} x^{\lambda} x^{\sigma}$. Here $a_{\mu\nu\lambda}$ is a constant antisymmetric tensor. When $f_{\mu\nu}$ is non-degenerate we can construct a non-degenerate Killing tensor. Let us consider for example the case of dimension $N=4$ and let $a_{\mu\nu\lambda}$ be a constant tensor having components ± 1 . The non-degenerate Killing-Yano tensor $f_{\mu\nu}$ has the following components

$$f_{12} = z+t, f_{13} = -y+t, f_{14} = -y-z, f_{23} = x+t, f_{24} = x-z, f_{34} = x+y. \tag{22}$$

Using (13) the corresponding non-degenerate Killing tensor has the components

$$\begin{aligned}
K_{11} &= (z+t)^2 + (-y+t)^2 + (-y-z)^2, K_{22} = (-z-t)^2 + (x+t)^2 + (x-z)^2 \\
K_{33} &= (y-t)^2 + (-x-t)^2 + (y+x)^2, K_{44} = (z+y)^2 + (-x+z)^2 + (-y-x)^2 \\
K_{12} &= (z+t)(x+t) + (-y-z)(x-z), K_{13} = (z+t)(-x+t) + (-y-z)(x+y) \\
K_{14} &= (z+t)(-x+z) + (-y+t)(-x-y), K_{23} = (-z-t)(y-t) + (t-z)(x+y) \\
K_{24} &= (-z-t)(y+z) + (x+t)(-x-y), \\
K_{34} &= (y-t)(y+z) + (-x-t)(-x+z).
\end{aligned} \tag{23}$$

It can be easily verified that the metric (23) is not an Einstein's metric. It has the scalar curvature

$$R = \frac{R_1}{R_2}, \tag{24}$$

where

$$\begin{aligned}
R_1 = & 42t^7 + 9x^5z^2 - 67xt^6 + 88t^3y^2xz + 24y^3x^2zt + 24ty^3z^2x + 22z^4x^3 \\
& + 9z^6x - 12x^4z^3 + 41z^3x^3t + 8z^2t^2x^2y - 10z^3t^2xy + 14z^2x^3yt + 24x^4yzt \\
& + 52t^3yx^2z - 276t^3yxz^2 - 136t^3z^4 - 167t^4z^3 + 46t^4z^2x + 18zx^5t + 12x^4z^2 \\
& + 22x^3z^2y^2 + 12x^2z^2y^3 + 9xz^2y^4 + 6z^6t - 86z^5t^2 - 296zt^4yx - 146t^5z^2 \\
& - 61t^6z + 198xz^4t^2 - 102x^4t^3 - 168x^2t^5 - 12x^2z^5 - 102x^3t^4 + 44yzt^5 - 36yt^5x \\
& + 30zt^5x + 9t^2x^5 + 18ty^4xz + 15ty^2x^2z^2 + 47ty^2xz^3 - 270t^2y^2x^2z + 105t^2y^2xz^2 \\
& + 44xz^3y^2t + 36z^3tx^2y + 23z^5tx + 145t^3z^3x + 9ty^4z^2 + 9t^2y^4x - 117t^2y^4z \\
& - 4z^3x^2y^2t + 4z^3x^3y + 4z^4x^2y + 22z^4xy^4 + 12z^5xy + 96t^3x^3z - 226t^3z^2 \\
& - 118t^4x^2z - 93t^2x^4z + 207t^2x^3z^2 - 6tx^4z^2 - 6z^4x^2t - 255z^3t^2x^2 \\
& + 12t^2x^4y - 144t^2z^4 - 56t^3yz^3 + 22t^2x^3y^2 + 19z^4ty^2 - 261z^3t^2y^2 - 164t^3yx^3 \\
& - 168y^3t^2z^2 - 180y^3t^3x - 12y^3t^3z - 233t^3y^2z^2 + 2t^4y^2x - 174t^4y^2z + 12ty^3z^3 \\
& + 12y^3z^3x + 12y^3t^2x^2 + 10z^5yt + 34z^2t^4y - 292x^2t^3y^2 + 76t^4x^2y + 24z^4txy \\
& - 152t^2x^3yz - 168y^3t^2xz + 168t^4y^3 - 126t^3y^4 - 224t^5y^2 + 112yt^6 \quad (25)
\end{aligned}$$

and

$$\begin{aligned}
R_2 = & (3z^2t^2 - 6zt^2x + 3t^2x^2 - 2x^2yt + 2z^3t + 4ztxy - 2z^2yt + 2x^3t - 2z^2xt - 2ztx^2 \\
& + 3x^2y^2 + 2yz^3 - 8xz^3 + 3z^2y^2 - 6xzy^2 + 3x^4 - 2x^2zy - 2z^2xy - 8x^3z + 2x^3y + 3z^4 \\
& + 10z^2x^2)(3x^2z + 3tx^2 + 2yxz + 2t^2x + 2yxt - 2xz^2 + 5zt^2 + 5z^2t + 3z^3 + 2yz^2 \\
& + 3t^3 + 3y^2z - 2t^2y + 3ty^2)(z + t)t^3. \quad (26)
\end{aligned}$$

3 Examples

3.1 Kerr-Newman geometry

The Kerr-Newman geometry describes a charged spinning black hole; in a standard choice of coordinates, the metric is given by the following line element [2]

$$ds^2 = -\frac{\Delta}{\rho^2} [dt - a \sin^2(\theta) d\varphi]^2 + \frac{\sin^2(\theta)}{\rho^2} [(r^2 + a^2) d\varphi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (27)$$

Here

$$\Delta = r^2 + a^2 - 2Mr + Q^2, \rho^2 = r^2 + a^2 \cos^2 \theta. \quad (28)$$

with Q the background electric charge, and $J = Ma$ the total angular momentum. The expression for ds^2 only describes the fields *outside* the horizon, which is located at

$$r = M + (M^2 - Q^2 - a^2)^{1/2}. \quad (29)$$

The Killing- Yano tensor for the Kerr–Newman is defined by [2]

$$\begin{aligned} \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu &= a \cos \theta dr \wedge (dt - a \sin^2 \theta d\phi) \\ &+ r \sin \theta d\theta \wedge [-a dt + (r^2 + a^2) d\phi]. \end{aligned} \quad (30)$$

The Kerr-Newman metric admits a second-rank Killing tensor field. It can be described in this coordinate system by the quadratic form

$$\begin{aligned} dk^2 &= K_{\mu\nu} dx^\mu dx^\nu = \frac{a^2 \cos^2 \theta \Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \\ &\frac{r^2 \sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 - \frac{\rho^2}{\Delta a^2 \cos^2 \theta} dr^2 + \frac{\rho^2}{r^2} d\theta^2. \end{aligned} \quad (31)$$

Kerr-Newman metric admits two Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$. Using Proposition 1 we found that Kerr-Newman metric and its dual (31) have the same Killing vectors.

Solving Killing-Yano equations (12) we found no solution on the dual manifold. The dual metric (31) has scalar curvature

$$\begin{aligned} R &= -2 \frac{\cos^2 \theta r^4 - 5r^4 + 2 \cos^2 \theta M r^3 + r^2 a^2 \cos^4 \theta - r^2 \cos^2 \theta Q^2 - 6a^2 \cos^4 \theta M r}{(r^2 + a^2 \cos^2 \theta) \cos^2 \theta r^2} \\ &- \frac{10a^2 \cos^4 \theta Q^2 + 10a^4 \cos^4 \theta}{(r^2 + a^2 \cos^2 \theta) \cos^2 \theta r^2} \end{aligned} \quad (32)$$

The existence of Killing-Yano of valence three is an interesting question for Kerr-Newmann metric and its dual.

A tensor $f_{\mu_1 \mu_2 \mu_3}$ is called a Killing-Yano tensor of valence 3 if it is totally antisymmetric and satisfies the equation

$$f_{\mu_1 \mu_2 \mu_3; \lambda} + f_{\lambda \mu_2 \mu_3; \mu_1} = 0, \quad (33)$$

where comma denotes the covariant derivative.

On the other hand we know that for every Killing-Yano tensor $f_{\mu_1 \mu_2 \mu_3}$ we have new supersymmetric charges (for more details see Ref.[14]). In our case we have four independent components of $f_{\mu_1 \mu_2 \mu_3}$ namely, $f_{r\theta\phi}$, $f_{r\theta t}$, $f_{\theta\phi t}$, $f_{r\phi t}$ and 15 independent Killing-Yano equations. After some calculations we found that (33), has no solution for (27). Solving dual Killing-Yano equations (12) we found no solution for (31).

3.2 Separable coordinate systems in 1 + 1-dimensional Minkowski space

Separable orthogonal coordinate systems on n -dimensional manifolds are characterized by Stäckel systems, which is a system of n linearly independent

Killing tensors K_{ij} of order two, including the metric tensor [10]. Including the Cartesian system there are 10 orthogonal coordinate systems in 1 + 1-dimensional flat space such that the Klein-Gordon equation may be separated. In all of them coordinate lines are either straight lines or conic sections, the latter ones being arranged in one or two confocal families. In the following list the curvilinear coordinates are denoted by μ and ν . We listed below those coordinates systems we have used in this paper (for more details see for example Refs.[11],[12]).

1. Cartesian system: Coordinates t and x .
2. Elliptic system: Elliptic coordinates μ and ν are defined by

$$t^2 = \mu\nu, \quad x^2 = (1 - \mu)(1 - \nu), \quad 0 < \nu, \mu < 1. \quad (34)$$

with μ and ν labeling ellipses of one and the same confocal family with mutually orthogonal intersections, given by the equations

$$\frac{t^2}{\mu} + \frac{x^2}{1 - \mu} = 1. \quad (35)$$

This system is defined in the square $|t| + |x| \leq 1$.

3. Hyperbolic system: Defined by the same equations as the elliptic ones, but with $1 < \nu < \mu < \infty$, so that (35) describes one family of hyperbolas. This hyperbolic system which is a continuation of the elliptic one to other domains in space-time, it may be defined in 4 wedges of space-time, $|t| - |x| > 1$ or $|x| - |t| > 1$.

Killing tensor K_{ik} associated with this systems is constructed from the flat metric in elliptic (hyperbolic) coordinates

$$ds^2 = \frac{\mu - \nu}{4} \left(\frac{d\mu^2}{\mu(\mu - 1)} - \frac{d\nu^2}{\nu(\nu - 1)} \right). \quad (36)$$

and has the following form [12]

$$K_{ik} = \frac{\mu - \nu}{4} \begin{pmatrix} \frac{1}{\mu(\nu+1)(\mu-1)} & 0 \\ 0 & \frac{-1}{\nu(\nu-1)(\mu+1)} \end{pmatrix}. \quad (37)$$

Taking into account (37) we found that the dual metric has the form

$$ds^2 = \frac{\mu - \nu}{4} \left(\frac{d\mu^2}{\mu(\nu + 1)(\mu - 1)} - \frac{d\nu^2}{\nu(\nu - 1)(\mu + 1)} \right). \quad (38)$$

Solving Killing-Yano equations corresponding to (36) we get

$$f_{\mu\nu} = -\frac{(\mu - \nu)^2}{16\mu\nu(\mu - 1)(\nu - 1)}. \quad (39)$$

We found that dual Killing-Yano equations have the following solution

$$f_{\mu\nu} = -\frac{(\mu - \nu)^2}{16\mu\nu(\mu - 1)(\nu - 1)(\nu + 1)(\mu + 1)}. \quad (40)$$

4 Conclusions

In this paper the geometric duality between local geometry described by $g_{\mu\nu}$ and the local geometry described by Killing tensor $K_{\mu\nu}$ were investigated. We found that the transformation (3) is not a conformal transformation. The Killing vectors equations and the Killing-Yano equations were analyzed on the dual manifold. When $D_\lambda K_{\mu\nu} = 0$ the symmetries of manifold and its dual coincide. We found the equations satisfied by the dual metric $K_{\mu\nu}$. It was found that if $g_{\mu\nu}$ satisfies Einstein's equations in vacuum the corresponding dual manifold $K_{\mu\nu}$ is not an Einstein's metric. We have proved that the dual Kerr-Newman metric admits the same Killing vectors as Kerr-Newman metric but it has no Killing-Yano tensor of order two and three. In the case of the separable coordinates in $1 + 1$ dimensions the corresponding metric and the dual metric have Killing-Yano tensors and the same Killing vectors. The classification of all Riemannian manifolds admitting a non-degenerate Killing tensor is an interesting problem and it is under investigation [15].

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